Quick Quiz Sections 3.4–3.6

1. B. $y = \sin^4 u \qquad u = 3x$

$$\frac{dy}{du} = 4\sin^3 u \cos u \qquad \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = 12\sin^3(3x)\cos(3x)$$

2. A. $y = \cos x + \tan x$

$$y' = -\sin x + \sec^2 x$$

$$y'' = -\cos x + 2\sec^2 x \tan x$$

3. C. $x = 3\sin t \qquad y = 2\cos t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt} = \frac{d}{dt}(2\cos t) = -2\sin t$$

$$\frac{dx}{dt} = \frac{d}{dt}(3\sin t) = 3\cos t$$

$$\frac{dy}{dx} = \frac{-2\sin t}{3\cos t} = -\frac{2}{3}\tan t.$$

4. (a) $s(0) = -0^2 + 0 + 2 = 2$ m

(b) $v(t) = \frac{ds}{dt} = \frac{d}{dt}(-t^2 + t + 2)$
$= -2t + 1$ m/s

(c) When $v(t) > 0$

$$-2t + 1 > 0$$

$$t < \frac{1}{2}$$

$$0 \le t < \frac{1}{2} \quad \text{(time must be} \ge 0\text{)}$$

(d) $a(t) = \frac{dv}{dt} = \frac{d}{dt}(-2t + 1) = -2$ m/s^2

(e) $s(t) = 0$

$$-t^2 + t + 2 = 0$$

$$-(t^2 - t - 2) = 0$$

$$-(t - 2)(t + 1) = 0$$

$t = 2$ or $t = -1$ (not in domain)

speed $= |v(2)| = |-2(2) + 1| = 3$ m/s

Section 3.7 Implicit Differentiation (pp. 157–164)

Exploration 1 An Unexpected Derivative

1. $2x - 2y - 2xy' + 2yy' = 0$. Solving for y', we find that $\frac{dy}{dx} = 1$ (provided $y \ne x$).

2. With a constant derivative of 1, the graph would seem to be a line with slope 1.

3. Letting $x = 0$ in the original equation, we find that $y = \pm 2$. This would seem to indicate that this equation defines two lines implicitly, both with slope 1. The two lines are $y = x + 2$ and $y = x - 2$.

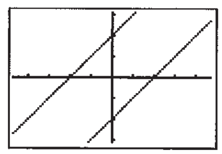
4. Factoring the original equation, we have

$$[(x - y) - 2][(x - y) + 2] = 0$$

$$\therefore x - y - 2 = 0 \text{ or } x - y + 2 = 0$$

$$\therefore y = x - 2 \text{ or } y = x + 2.$$

The graph is shown below.



[−4.7, 4.7] by [−3.1, 3.1]

5. At each point (x, y) on either line, $\frac{dy}{dx} = 1$. The condition $y \ne x$ is true because both lines are parallel to the line $y = x$. The derivative is surprising because it does not depend on x or y, but there are no inconsistencies.

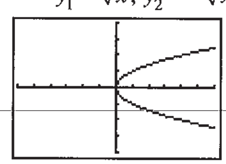
Quick Review 3.7

1. $x - y^2 = 0$

$$x = y^2$$

$$\pm\sqrt{x} = y$$

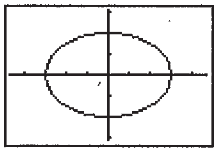
$$y_1 = \sqrt{x},\ y_2 = -\sqrt{x}$$



[-6, 6] by [-4, 4]

2. $4x^2+9y^2=36$

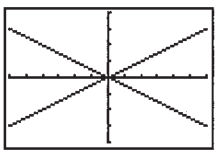
$$9y^2=36-4x^2$$
$$y^2=\frac{36-4x^2}{9}=\frac{4}{9}(9-x^2)$$
$$y=\pm\frac{2}{3}\sqrt{9-x^2}$$
$$y_1=\frac{2}{3}\sqrt{9-x^2},\ y_2=-\frac{2}{3}\sqrt{9-x^2}$$



[-4.7, 4.7] by [-3.1, 3.1]

3. $x^2-4y^2=0$

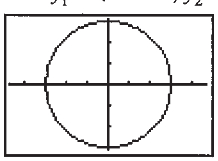
$$(x+2y)(x-2y)=0$$
$$y=\pm\frac{x}{2}$$
$$y_1=\frac{x}{2},\ y_2=-\frac{x}{2}$$



[-6, 6] by [-4, 4]

4. $x^2+y^2=9$

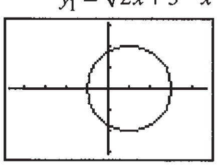
$$y^2=9-x^2$$
$$y=\pm\sqrt{9-x^2}$$
$$y_1=\sqrt{9-x^2},\ y_2=-\sqrt{9-x^2}$$



[-4.7, 4.7] by [-3.1, 3.1]

5. $x^2+y^2=2x+3$

$$y^2=2x+3-x^2$$
$$y=\pm\sqrt{2x+3-x^2}$$
$$y_1=\sqrt{2x+3-x^2},\ y_2=-\sqrt{2x+3-x^2}$$



[-4.7, 4.7] by [-3.1, 3.1]

6. $x^2y'-2xy=4x-y$

$$x^2y'=4x-y+2xy$$
$$y'=\frac{4x-y+2xy}{x^2}$$

7. $y'\sin x-x\cos x=xy'+y$

$$y'\sin x-xy'=y+x\cos x$$
$$(\sin x-x)y'=y+x\cos x$$
$$y'=\frac{y+x\cos x}{\sin x-x}$$

8. $x(y^2-y')=y'(x^2-y)$

$$xy^2=y'(x^2-y+x)$$
$$y'=\frac{xy^2}{x^2-y+x}$$

9. $\sqrt{x}(x-\sqrt[3]{x})=x^{1/2}(x-x^{1/3})$

$$=x^{1/2}x-x^{1/2}x^{1/3}$$
$$=x^{3/2}-x^{5/6}$$

10. $\dfrac{x+\sqrt[3]{x^2}}{\sqrt{x^3}}=\dfrac{x+x^{2/3}}{x^{3/2}}$

$$=\frac{x}{x^{3/2}}+\frac{x^{2/3}}{x^{3/2}}$$
$$=x^{-1/2}+x^{-5/6}$$

Section 3.7 Exercises

1. $x^2y+xy^2=6$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(xy^2)=\frac{d}{dx}(6)$$
$$x^2\frac{dy}{dx}+y(2x)+x(2y)\frac{dy}{dx}+y^2(1)=0$$
$$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}=-(2xy+y^2)$$
$$(2xy+x^2)\frac{dy}{dx}=-(2xy+y^2)$$
$$\frac{dy}{dx}=-\frac{2xy+y^2}{2xy+x^2}$$

2. $x^3+y^3=18xy$

$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)=\frac{d}{dx}(18xy)$$
$$3x^2+3y^2\frac{dy}{dx}=18x\frac{dy}{dx}+18y(1)$$
$$3y^2\frac{dy}{dx}-18x\frac{dy}{dx}=18y-3x^2$$
$$(3y^2-18x)\frac{dy}{dx}=18y-3x^2$$
$$\frac{dy}{dx}=\frac{18y-3x^2}{3y^2-18x}$$
$$\frac{dy}{dx}=\frac{6y-x^2}{y^2-6x}$$

3. $y^2 = \frac{x-1}{x+1}$

$$\frac{d}{dx}y^2 = \frac{d}{dx}\frac{x-1}{x+1}$$
$$2y\frac{dy}{dx} = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$
$$2y\frac{dy}{dx} = \frac{2}{(x+1)^2}$$
$$\frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

4. $x^2 = \frac{x-y}{x+y}$

$$\frac{d}{dx}(x^2) = \frac{d}{dx}\frac{x-y}{x+y}$$
$$2x = \frac{(x+y)\left(1-\frac{dy}{dx}\right)-(x-y)\left(1+\frac{dy}{dx}\right)}{(x+y)^2}$$
$$2x = \frac{\left[x-x\frac{dy}{dx}+y-y\frac{dy}{dx}\right]-\left[x+x\frac{dy}{dx}-y-y\frac{dy}{dx}\right]}{(x+y)^2}$$
$$2x = \frac{2y-2x\frac{dy}{dx}}{(x+y)^2}$$
$$x(x+y)^2 = y-x\frac{dy}{dx}$$
$$x\frac{dy}{dx} = y-x(x+y)^2$$
$$\frac{dy}{dx} = \frac{y-x(x+y)^2}{x} = \frac{y}{x}-(x+y)^2$$

Alternate solution:

$$x^2 = \frac{x-y}{x+y}$$
$$x^2(x+y) = x-y$$
$$x^3+x^2y = x-y$$
$$\frac{d}{dx}(x^3)+\frac{d}{dx}(x^2y) = \frac{d}{dx}(x)-\frac{d}{dx}(y)$$
$$3x^2+x^2\frac{dy}{dx}+y(2x) = 1-\frac{dy}{dx}$$
$$(x^2+1)\frac{dy}{dx} = 1-3x^2-2xy$$
$$\frac{dy}{dx} = \frac{1-3x^2-2xy}{x^2+1}$$

5. $x = \tan y$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$
$$1 = \sec^2 y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

6. $x = \sin y$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$
$$1 = \cos y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cos y} = \sec y$$

7. $x + \tan xy = 0$

$$\frac{d}{dx}(x)+\frac{d}{dx}(\tan xy) = \frac{d}{dx}(0)$$
$$1+\sec^2(xy)\frac{d}{dx}(xy) = 0$$
$$1+(\sec^2 xy)[x\frac{dy}{dx}+(y)\,(1)] = 0$$
$$(\sec^2 xy)\,(x)\frac{dy}{dx} = -1-(\sec^2 xy)(y)$$
$$\frac{dy}{dx} = \frac{-1-y\sec^2 xy}{x\sec^2 xy}$$
$$\frac{dy}{dx} = -\frac{1}{x}\cos^2 xy-\frac{y}{x}$$

8. $x + \sin y = xy$

$$\frac{d}{dx}(x)+\frac{d}{dx}(\sin y) = \frac{d}{dx}(xy)$$
$$1+(\cos y)\frac{dy}{dx} = x\frac{dy}{dx}+(y)(1)$$
$$(\cos y-x)\frac{dy}{dx} = -1+y$$
$$\frac{dy}{dx} = \frac{-1+y}{\cos y-x} = \frac{1-y}{x-\cos y}$$

9. $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(13)$

$$2x+2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y},\ -\frac{-2}{3} = \frac{2}{3}$$

10. $\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(9)$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y},\ -\frac{0}{3}=0$$

11. $\frac{d}{dx}((x-1)^2+(y-1)^2)=\frac{d}{dx}(13)$

$$2(x-1)1+(2(y-1)1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x-1}{y-1},-\frac{3-1}{4-1}=-\frac{2}{3}$$

12. $\frac{d}{dx}((x+2)^2+(y+3)^2)=\frac{d}{dx}(25)$

$$2(x+2)1+(2(y+3)1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x+2}{y+3},-\frac{1+2}{-7+3}=\frac{3}{4}$$

13. $\frac{d}{dx}(x^2y-xy^2)=\frac{d}{dx}(4)$

$$x^2\frac{dy}{dx}+y\cdot 2x-\left(x\cdot 2y\frac{dy}{dx}+y^2\right)=0$$

$$(x^2-2xy)\frac{dy}{dx}+2xy-y^2=0$$

$$(x^2-2xy)\frac{dy}{dx}=y^2-2xy$$

$$\frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$$

defined at every point except where $x=0$ or $y=\frac{x}{2}$.

14. $\frac{d}{dx}(x)=\frac{d}{dx}(\cos y)$

$$1=-\sin y\frac{dy}{dx}$$

$$\frac{dy}{dx}=-\frac{1}{\sin y},$$

is defined everywhere except where $\sin y=0$:

$$y=\pm k\pi$$

$$x=\cos(k\pi)$$

$$x=1 \quad\text{or}\quad -1$$

15. $\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(xy)$

$$3x^2+3y^2\frac{dy}{dx}=y+x\frac{dx}{dy}$$

$$3x^2-y=(x-3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{3x^2-y}{x-3y^2},$$

defined everywhere except where $y^2=\frac{x}{3}$

16. $\frac{d}{dx}(x^2+4xy+4y^2-3x)=\frac{d}{dx}(6)$

$$2x+4y-3+(4x+8y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{3-2x-4y}{4x+8y},$$

defined everywhere except where $y=-\frac{1}{2}x$

17. $x^2+xy-y^2=1$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(xy)-\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$$

$$2x+x\frac{dy}{dx}+(y)(1)-2y\frac{dy}{dx}=0$$

$$(x-2y)\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}=\frac{-2x-y}{x-2y}=\frac{2x+y}{2y-x}$$

Slope at (2, 3): $\frac{2(2)+3}{2(3)-2}=\frac{7}{4}$

(a) Tangent: $y=\frac{7}{4}(x-2)+3$ or $y=\frac{7}{4}x-\frac{1}{2}$

(b) Normal: $y=-\frac{4}{7}(x-2)+3$ or $y=-\frac{4}{7}x+\frac{29}{7}$

18. $x^2+y^2=25$

$$\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)=\frac{d}{dx}(25)$$

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

Slope at $(3,-4)$: $-\frac{3}{-4}=\frac{3}{4}$

(a) Tangent: $y=\frac{3}{4}(x-3)+(-4)$ or $y=\frac{3}{4}x-\frac{25}{4}$

(b) Normal: $y=-\frac{4}{3}(x-3)+(-4)$ or $y=-\frac{4}{3}x$

19. $x^2y^2=9$

$$\frac{d}{dx}(x^2y^2)=\frac{d}{dx}(9)$$

$$(x^2)(2y)\frac{dy}{dx}+(y^2)(2x)=0$$

$$2x^2y\frac{dy}{dx}=-2xy^2$$

$$\frac{dy}{dx}=-\frac{2xy^2}{2x^2y}=-\frac{y}{x}$$

Slope at (−1, 3): $-\frac{3}{-1}=3$

(a) Tangent: $y=3(x+1)+3$ or $y=3x+6$

(b) Normal: $y=-\frac{1}{3}(x+1)+3$ or $y=-\frac{1}{3}x+\frac{8}{3}$

20.
$$y^2-2x-4y-1=0$$
$$\frac{d}{dx}(y^2)-\frac{d}{dx}(2x)-\frac{d}{dx}(4y)-\frac{d}{dx}(1)=\frac{d}{dx}(0)$$
$$2y\frac{dy}{dx}-2-4\frac{dy}{dx}-0=0$$
$$(2y-4)\frac{dy}{dx}=2$$
$$\frac{dy}{dx}=\frac{1}{y-2}$$

Slope at $(-2, 1)$: $\frac{1}{1-2}=-1$

(a) Tangent: $y=-(x+2)+1$ or $y=-x-1$

(b) Normal: $y=1(x+2)+1$ or $y=x+3$

21.
$$6x^2+3xy+2y^2+17y-6=0$$
$$\frac{d}{dx}(6x^2)+\frac{d}{dx}(3xy)+\frac{d}{dx}(2y^2)+\frac{d}{dx}(17y)-\frac{d}{dx}(6)=\frac{d}{dx}(0)$$
$$12x+3x\frac{dy}{dx}+(3y)(1)+4y\frac{dy}{dx}+17\frac{dy}{dx}-0=0$$
$$3x\frac{dy}{dx}+4y\frac{dy}{dx}+17\frac{dy}{dx}=-12x-3y$$
$$(3x+4y+17)\frac{dy}{dx}=-12x-3y$$
$$\frac{dy}{dx}=\frac{-12x-3y}{3x+4y+17}$$

Slope at $(-1, 0)$: $\frac{-12(-1)-3(0)}{3(-1)+4(0)+17}=\frac{12}{14}=\frac{6}{7}$

(a) Tangent: $y=\frac{6}{7}(x+1)+0$ or $y=\frac{6}{7}x+\frac{6}{7}$

(b) Normal: $y=-\frac{7}{6}(x+1)+0$ or $y=-\frac{7}{6}x-\frac{7}{6}$

22.
$$x^2-\sqrt{3}xy+2y^2=5$$
$$\frac{d}{dx}(x^2)-\sqrt{3}\frac{d}{dx}(xy)+2\frac{d}{dx}(y^2)=\frac{d}{dx}(5)$$
$$2x-\sqrt{3}(x)\frac{dy}{dx}-\sqrt{3}(y)(1)+4y\frac{dy}{dx}=0$$
$$(-x\sqrt{3}+4y)\frac{dy}{dx}=y\sqrt{3}-2x$$
$$\frac{dy}{dx}=\frac{y\sqrt{3}-2x}{-x\sqrt{3}+4y}$$

Slope at $(\sqrt{3}, 2)$: $\frac{2\sqrt{3}-2\sqrt{3}}{-\sqrt{3}\sqrt{3}+4(2)}=0$

(a) Tangent: $y=2$

(b) Normal: $x=\sqrt{3}$

23.
$$2xy+\pi\sin y=2\pi$$
$$2\frac{d}{dx}(xy)+\pi\frac{d}{dx}(\sin y)=\frac{d}{dx}(2\pi)$$
$$2x\frac{dy}{dx}+2y(1)+\pi\cos y\frac{dy}{dx}=0$$
$$(2x+\pi\cos y)\frac{dy}{dx}=-2y$$
$$\frac{dy}{dx}=-\frac{2y}{2x+\pi\cos y}$$

Slope at $\left(1, \frac{\pi}{2}\right)$: $-\frac{2(\pi/2)}{2(1)+\pi\cos(\pi/2)}=-\frac{\pi}{2}$

(a) Tangent: $y=-\frac{\pi}{2}(x-1)+\frac{\pi}{2}$ or $y=-\frac{\pi}{2}x+\pi$

(b) Normal: $y=\frac{2}{\pi}(x-1)+\frac{\pi}{2}$ or $y=\frac{2}{\pi}x-\frac{2}{\pi}+\frac{\pi}{2}$

24. $x\sin 2y=y\cos 2x$
$$\frac{d}{dx}(x\sin 2y)=\frac{d}{dx}(y\cos 2x)$$
$$(x)(\cos 2y)(2)\frac{dy}{dx}+(\sin 2y)(1)$$
$$=(y)(-\sin 2x)(2)+(\cos 2x)\left(\frac{dy}{dx}\right)$$
$$(2x\cos 2y)\frac{dy}{dx}-(\cos 2x)\frac{dy}{dx}=-2y\sin 2x-\sin 2y$$
$$\frac{dy}{dx}=-\frac{2y\sin 2x+\sin 2y}{2x\cos 2y-\cos 2x}$$

Slope at $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$: $-\frac{2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)+\sin(\pi)}{2\left(\frac{\pi}{4}\right)\cos(\pi)-\cos\left(\frac{\pi}{2}\right)}$
$$=-\frac{(\pi)(1)+0}{\left(\frac{\pi}{2}\right)(-1)-0}=2$$

(a) Tangent: $y=2\left(x-\frac{\pi}{4}\right)+\frac{\pi}{2}$ or $y=2x$

(b) Normal: $y=-\frac{1}{2}\left(x-\frac{\pi}{4}\right)+\frac{\pi}{2}$ or $y=-\frac{1}{2}x+\frac{5\pi}{8}$

25.
$$y=2\sin(\pi x-y)$$
$$\frac{dy}{dx}=\frac{d}{dx}2\sin(\pi x-y)$$
$$\frac{dy}{dx}=2\cos(\pi x-y)\left(\pi-\frac{dy}{dx}\right)$$
$$[1+2\cos(\pi x-y)]\frac{dy}{dx}=2\pi\cos(\pi x-y)$$
$$\frac{dy}{dx}=\frac{2\pi\cos(\pi x-y)}{1+2\cos(\pi x-y)}$$

Slope at $(1, 0)$: $\frac{2\pi\cos\pi}{1+2\cos\pi}=\frac{2\pi(-1)}{1+2(-1)}=2\pi$

(a) Tangent: $y=2\pi(x-1)+0$ or $y=2\pi x-2\pi$

(b) Normal: $y=-\frac{1}{2\pi}(x-1)+0$ or $y=-\frac{x}{2\pi}+\frac{1}{2\pi}$

26. $x^2\cos^2 y - \sin y = 0$

$$\frac{d}{dx}(x^2\cos^2 y) - \frac{d}{dx}(\sin y) = \frac{d}{dx}(0)$$

$$(x^2)(2\cos y)(-\sin y)\left(\frac{dy}{dx}\right) + (\cos^2 y)(2x) - (\cos y)\frac{dy}{dx} = 0$$

$$-(2x^2\cos y\sin y + \cos y)\frac{dy}{dx} = -2x\cos^2 y$$

$$\frac{dy}{dx} = \frac{2x\cos^2 y}{\cos y + 2x^2\cos y\sin y} = \frac{2x\cos y}{1+2x^2\sin y}$$

Slope at $(0, \pi)$: $\dfrac{2(0)\cos\pi}{1+2(0)^2\sin\pi} = 0$

(a) Tangent: $y = \pi$

(b) Normal: $x = 0$

27. $x^2 + y^2 = 1$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = -\frac{x}{y}$$

$$y'' = \frac{d}{dx}\left(-\frac{x}{y}\right)$$

$$= -\frac{(y)(1)-(x)(y')}{y^2}$$

$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$= -\frac{x^2+y^2}{y^3}$$

Since our original equation was $x^2 + y^2 = 1$, we may substitute 1 for $x^2 + y^2$, giving $y'' = -\dfrac{1}{y^3}$.

28. $x^{2/3} + y^{2/3} = 1$

$$\frac{d}{dx}(x^{2/3}) + \frac{d}{dx}(y^{2/3}) = \frac{d}{dx}(1)$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$

$$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = \frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{1/3}\right]$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\frac{xy'-(y)(1)}{x^2}$$

$$= -\frac{1}{3}\,\frac{-(x)\left(\frac{y}{x}\right)^{1/3} - y}{x^{4/3}y^{2/3}}$$

$$= \frac{1}{3}\,\frac{x^{2/3}y^{1/3}+y}{x^{4/3}y^{2/3}}$$

$$= \frac{x^{2/3}+y^{2/3}}{3x^{4/3}y^{1/3}}$$

Since our original equation was $x^{2/3} + y^{2/3} = 1$, we may substitute 1 for $x^{2/3} + y^{2/3}$, giving $y'' = \dfrac{1}{3x^{4/3}y^{1/3}}$.

29. $y^2 = x^2 + 2x$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x)$$

$$2yy' = 2x + 2$$

$$y' = \frac{2x+2}{2y} = \frac{x+1}{y}$$

$$y'' = \frac{d}{dx}\left(\frac{x+1}{y}\right)$$

$$= \frac{(y)(1)-(x+1)y'}{y^2}$$

$$= \frac{y-(x+1)\left(\frac{x+1}{y}\right)}{y^2}$$

$$= \frac{y^2-(x+1)^2}{y^3}$$

Since our original equation was $y^2 = x^2 + 2x$, we may write $y^2 - (x+1)^2 = (x^2+2x) - (x^2+2x+1) = -1$, which gives $y'' = -\dfrac{1}{y^3}$.

30. $y^2 + 2y = 2x + 1$

$$\frac{d}{dx}(y^2+2y) = \frac{d}{dx}(2x+1)$$

$$(2y+2)y' = 2$$

$$y' = \frac{1}{y+1}$$

30. Continued

$$y'' = \frac{d}{dx}\frac{1}{y+1}$$
$$= -(y+1)^{-2}y'$$
$$= -(y+1)^{-2}\left(\frac{1}{y+1}\right)$$
$$= -\frac{1}{(y+1)^3}$$

31. $\frac{dy}{dx} = \frac{d}{dx}x^{9/4} = \frac{9}{4}x^{(9/4)-1} = \frac{9}{4}x^{5/4}$

32. $\frac{dy}{dx} = \frac{d}{dx}x^{-3/5} = -\frac{3}{5}x^{(-3/5)-1} = -\frac{3}{5}x^{-8/5}$

33. $\frac{dy}{dx} = \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{(1/3)-1} = \frac{1}{3}x^{-2/3}$

34. $\frac{dy}{dx} = \frac{d}{dx}\sqrt[4]{x} = \frac{d}{dx}x^{1/4} = \frac{1}{4}x^{(1/4)-1} = \frac{1}{4}x^{-3/4}$

35. $\frac{dy}{dx} = \frac{d}{dx}(2x+5)^{-1/2} = -\frac{1}{2}(2x+5)^{(-1/2)-1}\frac{d}{dx}(2x+5)$

$$= -\frac{1}{2}(2x+5)^{-3/2}(2) = -(2x+5)^{-3/2}$$

36. $\frac{dy}{dx} = \frac{d}{dx}(1-6x)^{2/3}$

$$= \frac{2}{3}(1-6x)^{(2/3)-1}\frac{d}{dx}(1-6x)$$
$$= \frac{2}{3}(1-6x)^{-1/3}(-6)$$
$$= -4(1-6x)^{-1/3}$$

37. $\frac{dy}{dx} = \frac{d}{dx}\left(x\sqrt{x^2+1}\right)$

$$= x\frac{d}{dx}\sqrt{x^2+1} + \sqrt{x^2+1}\frac{d}{dx}(x)$$
$$= x\frac{d}{dx}(x^2+1)^{1/2} + (x^2+1)^{1/2}$$
$$= x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x) + (x^2+1)^{1/2}$$
$$= x^2(x^2+1)^{-1/2} + (x^2+1)^{1/2}$$

Note: This answer is equivalent to $\frac{2x^2+1}{\sqrt{x^2+1}}$.

38. $\frac{dy}{dx} = \frac{d}{dx}\frac{x}{\sqrt{x^2+1}} = \frac{(x^2+1)^{1/2}\frac{d}{dx}x - x\frac{d}{dx}(x^2+1)^{1/2}}{x^2+1}$

$$= \frac{(x^2+1)^{1/2} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$$
$$= \frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}}$$
$$= \frac{1}{(x^2+1)^{3/2}}$$
$$= (x^2+1)^{-3/2}$$

39. $\frac{dy}{dx} = \frac{d}{dx}(1-x^{1/2})^{1/2}$

$$= \frac{1}{2}(1-x^{1/2})^{-1/2}\frac{d}{dx}(1-x^{1/2})$$
$$= \frac{1}{2}(1-x^{1/2})^{-1/2}\left(-\frac{1}{2}x^{-1/2}\right)$$
$$= -\frac{1}{4}(1-x^{1/2})^{-1/2}x^{-1/2}$$

40. $\frac{dy}{dx} = \frac{d}{dx}3(2x^{-1/2}+1)^{-1/3}$

$$= -(2x^{-1/2}+1)^{-4/3}\frac{d}{dx}(2x^{-1/2}+1)$$
$$= -(2x^{-1/2}+1)^{-4/3}(-x^{-3/2})$$
$$= x^{-3/2}(2x^{-1/2}+1)^{-4/3}$$

41. $\frac{dy}{dx} = \frac{d}{dx}3(\csc x)^{3/2}$

$$= \frac{9}{2}(\csc x)^{1/2}\frac{d}{dx}(\csc x)$$
$$= \frac{9}{2}(\csc x)^{1/2}(-\csc x \cot x)$$
$$= -\frac{9}{2}(\csc x)^{3/2}\cot x$$

42. $\frac{dy}{dx} = \frac{d}{dx}[\sin(x+5)]^{5/4}$

$$= \frac{5}{4}[\sin(x+5)]^{1/4}\frac{d}{dx}\sin(x+5)$$
$$= \frac{5}{4}[\sin(x+5)]^{1/4}\cos(x+5)$$

43. (a) If $f(x) = \frac{3}{2}x^{2/3} - 3$, then

$f'(x) = x^{-1/3}$ and $f''(x) = -\frac{1}{3}x^{-4/3}$

which contradicts the given equation $f''(x) = x^{-1/3}$.

(b) If $f(x) = \frac{9}{10}x^{5/3} - 7$, then

$f'(x) = \frac{3}{2}x^{2/3}$ and $f''(x) = x^{-1/3}$, which matches the given equation.

(c) Differentiating both sides of the given equation

$f''(x) = x^{-1/3}$ gives $f'''(x) = -\frac{1}{3}x^{-4/3}$, so it *must* be true that $f'''(x) = -\frac{1}{3}x^{-4/3}$.

(d) If $f'(x) = \frac{3}{2}x^{2/3} + 6$, then $f''(x) = x^{-1/3}$, which matches the given equation.

Conclusion: (b), (c), and (d) could be true.

44. (a) If $g'(t) = 4\sqrt[4]{t} - 4$, then

$$g''(t) = \frac{d}{dx}(4t^{1/4} - 4) = t^{-3/4} = \frac{1}{t^{3/4}},$$ which matches the given equation.

(b) Differentiating both sides of the given equation $g''(t) = \frac{1}{t^{3/4}} = t^{-3/4}$ gives $g'''(t) = -\frac{3}{4}t^{-7/4}$, which is not consistent with $g'''(t) = -\frac{4}{\sqrt[4]{t}}$.

(c) If $g(t) = t - 7 + \frac{16}{5}t^{5/4}$, then

$g'(t) = 1 + 4t^{1/4}$ and $g''(t) = t^{-3/4} = \frac{1}{t^{3/4}}$, which matches the given equation.

(d) If $g'(t) = \frac{1}{4}t^{1/4}$, then $g''(t) = \frac{1}{16}t^{-3/4}$, which contradicts the given equation.

Conclusion: (a) and (c) could be true.

45. (a)

$$y^4 = y^2 - x^2$$

$$\frac{d}{dx}(y^4) = \frac{d}{dx}(y^2) - \frac{d}{dx}x^2$$

$$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$$

$$(4y^3 - 2y)\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y^3 - 2y} = \frac{x}{y - 2y^3}$$

At $\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$:

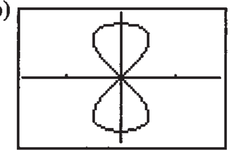
$$\text{Slope} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right)^3}$$

$$= \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{4}} \cdot \frac{\frac{4}{\sqrt{3}}}{\frac{4}{\sqrt{3}}} = \frac{1}{2-3} = -1$$

At $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$:

$$\text{Slope} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - 2\left(\frac{1}{2}\right)^3} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{1}{4}} \cdot \frac{4}{4} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

(b)



[−1.8, 1.8] by [−1.2, 1.2]

Parameter interval: $-1 \le t \le 1$

46. (a)

$$y^2(2 - x) = x^3$$

$$\frac{d}{dx}\left[y^2(2-x)\right] = \frac{d}{dx}(x^3)$$

$$(y^2)(-1) + (2-x)(2y)\frac{dy}{dx} = 3x^2$$

$$2y(2-x)\frac{dy}{dx} = 3x^2 + y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}$$

Slope at (1, 1): $\frac{3(1)^2 + (1)^2}{2(1)(2-1)} = \frac{4}{2} = 2$

Tangent: $y = 2(x - 1) + 1$ or $y = 2x - 1$

Normal: $y = -\frac{1}{2}(x-1) + 1$ or $y = -\frac{1}{2}x + \frac{3}{2}$

(b) One way is to graph the equations $y = \pm\sqrt{\frac{x^3}{2-x}}$.

47. (a) $(-1)^3(1)^2 = \cos(\pi)$ is true since both sides equal -1.

(b)

$$x^3y^2 = \cos(\pi y)$$

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}\cos(\pi y)$$

$$(x^3)(2y)\frac{dy}{dx} + (y^2)(3x^2) = (-\sin\pi y)(\pi)\frac{dy}{dx}$$

$$(2x^3y + \pi\sin\pi y)\frac{dy}{dx} = -3x^2y^2$$

$$\frac{dy}{dx} = -\frac{3x^2y^2}{2x^3y + \pi\sin\pi y}$$

Slope at $(-1, 1)$: $-\frac{3(-1)^2(1)}{2(-1)^3(1) + \pi\sin\pi} = \frac{-3}{-2} = \frac{3}{2}$

The slope of the tangent line is $\frac{3}{2}$.

48. (a) When $x = 2$, we have $y^3 - 2y = -1$, or $y^3 - 2y + 1 = 0$. Clearly, $y = 1$ is one solution, and we may factor $y^3 - 2y + 1$ as $(y-1)(y^2 + y - 1)$. The solutions of $y^2 + y - 1 = 0$ are $y = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$.

Hence, there are three possible y-values: 1, $\frac{-1-\sqrt{5}}{2}$, and $\frac{-1+\sqrt{5}}{2}$.

48. Continued

(b)
$$y^3 - xy = -1$$
$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) = \frac{d}{dx}(-1)$$
$$3y^2y' - xy' - (y)(1) = 0$$
$$(3y^2 - x)y' = y$$
$$y' = \frac{y}{3y^2 - x}$$
$$y'' = \frac{d}{dx}\frac{y}{3y^2 - x}$$
$$= \frac{(3y^2 - x)(y') - (y)(6yy' - 1)}{(3y^2 - x)^2}$$
$$= \frac{y - xy' - 3y^2y'}{(3y^2 - x)^2}$$

Since we are working with numerical information, there is no need to write a general expression for y'' in terms of x and y.

To evaluate $f'(2)$, evaluate the expression for y' using $x = 2$ and $y = 1$:
$$f'(2) = \frac{1}{3(1)^2 - 2} = \frac{1}{1} = 1$$

To evaluate $f''(2)$, evaluate the expression for y'' using $x = 2$, $y = 1$, and $y' = 1$:
$$f''(2) = \frac{(1) - 2(1) - 3(1)^2(1)}{[3(1)^2 - 2]^2} = \frac{-4}{1} = -4$$

49. Find the two points:

The curve crosses the x-axis when $y = 0$, so the equation becomes $x^2 + 0x + 0 = 7$, or $x^2 = 7$. The solutions are $x = \pm\sqrt{7}$, so the points are $(\pm\sqrt{7}, 0)$.

Show tangents are parallel:
$$x^2 + xy + y^2 = 7$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$
$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$
$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$
$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Slope at $(\sqrt{7}, 0)$: $-\frac{2(\sqrt{7}) + 0}{\sqrt{7} + 2(0)} = -2$

Slope at $(-\sqrt{7}, 0)$: $-\frac{2(-\sqrt{7}) + 0}{-\sqrt{7} + 2(0)} = -2$

The tangents at these points are parallel because they have the same slope. The common slope is -2.

50.
$$x^2 + xy + y^2 = 7$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7)$$
$$2x + x\frac{dy}{dx} + (y)(1) + 2y\frac{dy}{dx} = 0$$
$$(x + 2y)\frac{dy}{dx} = -(2x + y)$$
$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

(a) The tangent is parallel to the x-axis when $\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = 0$, or $y = -2x$.

Substituting $-2x$ for y in the original equation, we have
$$x^2 + xy + y^2 = 7$$
$$x^2 + (x)(-2x) + (-2x)^2 = 7$$
$$x^2 - 2x^2 + 4x^2 = 7$$
$$3x^2 = 7$$
$$x = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right)$ and $\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right)$.

(b) Since x and y are interchangeable in the original equation, $\frac{dx}{dy}$ can be obtained by interchanging x and y in the expression for $\frac{dy}{dx}$. That is, $\frac{dx}{dy} = -\frac{2y + x}{y + 2x}$. The tangent is parallel to the y-axis when $\frac{dx}{dy} = 0$, or $x = -2y$. Substituting $-2y$ for x in the original equation, we have:
$$x^2 + xy + y^2 = 7$$
$$(-2y)^2 + (-2y)(y) + y^2 = 7$$
$$4y^2 - 2y^2 + y^2 = 7$$
$$3y^2 = 7$$
$$y = \pm\sqrt{\frac{7}{3}}$$

The points are $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$ and $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$.

Note that these are the same points that would be obtained by interchanging x and y in the solution to part (a).

51. First curve:
$$2x^2 + 3y^2 = 5$$
$$\frac{d}{dx}(2x^2) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(5)$$
$$4x + 6y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{4x}{6y} = -\frac{2x}{3y}$$

51. Continued

Second curve:

$$y^2 = x^3$$
$$\frac{d}{dx}y^2 = \frac{d}{dx}x^3$$
$$2y\frac{dy}{dx} = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

At (1, 1), the slopes are $-\frac{2}{3}$ and $\frac{3}{2}$ respectively.

At (1, −1), the slopes are $\frac{2}{3}$ and $-\frac{3}{2}$ respectively. In both cases, the tangents are perpendicular. To graph the curves and normal lines, we may use the following parametric equations for $-\pi \le t \le \pi$:

First curve: $x = \sqrt{\frac{5}{2}}\cos t,\ y = \sqrt{\frac{5}{3}}\sin t$

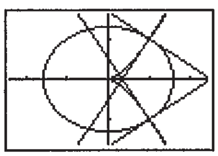
Second curve: $x = \sqrt[3]{t^2},\ y = t$

Tangents at (1, 1): $x = 1 + 3t, y = 1 - 2t$

$x = 1 + 2t,\ y = 1 + 3t$

Tangents at (1, −1): $x = 1 + 3t, y = -1 + 2t$

$x = 1 + 2t,\ y = -1 - 3t$



[−2.4, 2.4] by [−1.6, 1.6]

52. $v(t) = s'(t) = \frac{d}{dt}(4+6t)^{3/2} = \frac{3}{2}(4+6t)^{1/2}(6)$

$= 9(4+6t)^{1/2}$

$a(t) = v'(t) = \frac{d}{dt}[9(4+6t)^{1/2}]$

$= \frac{9}{2}(4+6t)^{-1/2}(6) = 27(4+6t)^{-1/2}$

At $t = 2$, the velocity is $v(2) = 36$ m/sec and the acceleration is $a(2) = \frac{27}{4}\text{ m/sec}^2$.

53. Acceleration $= \frac{dv}{dt} = \frac{d}{dt}[8(s-t)^{1/2} + 1]$

$= 4(s-t)^{-1/2}\left(\frac{ds}{dt} - 1\right)$

$= 4(s-t)^{-1/2}\ (v-1)$

$= 4(s-t)^{-1/2}\ [(8(s-t)^{1/2} + 1) - 1]$

$= 32(s-t)^{-1/2}\ (s-t)^{1/2}$

$= 32\text{ ft/sec}^2$

54.

$$y^4 - 4y^2 = x^4 - 9x^2$$
$$\frac{d}{dx}(y^4) - \frac{d}{dx}(4y^2) = \frac{d}{dx}(x^4) - \frac{d}{dx}(9x^2)$$
$$4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 18x$$
$$\frac{dy}{dx} = \frac{4x^3 - 18x}{4y^3 - 8y} = \frac{2x^3 - 9x}{2y^3 - 4y}$$

Slope at (3,2): $\frac{2(3)^3 - 9(3)}{2(2)^3 - 4(2)} = \frac{27}{8}$

Slope at (−3,2): $\frac{2(-3)^3 - 9(-3)}{2(2)^3 - 4(2)} = -\frac{27}{8}$

Slope at (−3,−2): $\frac{2(-3)^3 - 9(-3)}{2(-2)^3 - 4(-2)} = \frac{27}{8}$

Slope at (3,−2): $\frac{2(3)^3 - 9(3)}{2(-2)^3 - 4(-2)} = -\frac{27}{8}$

55. (a)

$$x^3 + y^3 - 9xy = 0$$
$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - 9\frac{d}{dx}(xy) = \frac{d}{dx}(0)$$
$$3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9(y)(1) = 0$$
$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$
$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

Slope at (4, 2): $\frac{3(2) - (4)^2}{(2)^2 - 3(4)} = \frac{-10}{-8} = \frac{5}{4}$

Slope at (2, 4): $\frac{3(4) - (2)^2}{(4)^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$

(b) The tangent is horizontal when

$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x} = 0$, or $y = \frac{x^2}{3}$.

Substituting $\frac{x^2}{3}$ for y in the original equation, we have:

$$x^3 + y^3 - 9xy = 0$$
$$x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0$$
$$x^3 + \frac{x^6}{27} - 3x^3 = 0$$
$$\frac{x^3}{27}(x^3 - 54) = 0$$
$$x = 0 \text{ or } x = \sqrt[3]{54} = 3\sqrt[3]{2}$$

At $x = 0$, we have $y = \frac{0^2}{3} = 0$, which gives the point (0, 0), which is the origin. At $x = 3\sqrt[3]{2}$, we have $y = \frac{1}{3}(3\sqrt[3]{2})^2 = \frac{1}{3}(9\sqrt[3]{4}) = 3\sqrt[3]{4}$, so the point other than the origin is $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ or approximately (3.780, 4.762).

55. Continued

(c) The equation $x^3 + y^3 - 9xy$ is not affected by interchanging x and y, so its graph is symmetric about the line $y = x$ and we may find the desired point by interchanging the x-value and the y-value in the answer to part (b). The desired point is $(3\sqrt[3]{4}, 3\sqrt[3]{2})$ or approximately (4.762, 3.780).

56.

$$x^2 + 2xy - 3y^2 = 0$$
$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(0)$$
$$2x + 2x\frac{dy}{dx} + 2(y)(1) - 6y\frac{dy}{dx} = 0$$
$$(2x - 6y)\frac{dy}{dx} = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 6y} = \frac{x + y}{3y - x}$$

At (1, 1) the curve has slope $\frac{1+1}{3(1)-1} = \frac{2}{2} = 1$, so the normal line is $y = -1(x - 1) + 1$ or $y = -x + 2$.

Substituting $-x + 2$ for y in the original equation, we have:

$$x^2 + 2xy - 3y^2 = 0$$
$$x^2 + 2x(-x + 2) - 3(-x + 2)^2 = 0$$
$$x^2 - 2x^2 + 4x - 3(x^2 - 4x + 4) = 0$$
$$-4x^2 + 16x - 12 = 0$$
$$-4(x - 1)(x - 3) = 0$$
$$x = 1 \text{ or } x = 3$$

Since the given point (1,1) had $x = 1$, we choose $x = 3$ and so $y = -(3) + 2 = -1$. The desired point is (3, −1).

57.

$$xy + 2x - y = 0$$
$$\frac{d}{dx}(xy) + \frac{d}{dx}(2x) - \frac{d}{dx}(y) = \frac{d}{dx}(0)$$
$$x\frac{dy}{dx} + (y)(1) + 2 - \frac{dy}{dx} = 0$$
$$(x - 1)\frac{dy}{dx} = -2 - y$$
$$\frac{dy}{dx} = \frac{-2 - y}{x - 1} = \frac{2 + y}{1 - x}$$

Since the slope of the line $2x + y = 0$ is −2, we wish to find points where the normal has slope −2, that is, where the tangent has slope $\frac{1}{2}$. Thus, we have

$$\frac{2 + y}{1 - x} = \frac{1}{2}$$
$$2(2 + y) = 1 - x$$
$$4 + 2y = 1 - x$$
$$x = -2y - 3$$

Substituting $-2y - 3$ in the original equation, we have:

$$xy + 2x - y = 0$$
$$(-2y - 3)y + 2(-2y - 3) - y = 0$$
$$-2y^2 - 8y - 6 = 0$$
$$-2(y + 1)(y + 3) = 0$$
$$y = -1 \text{ or } y = -3$$

At $y = -1, x = -2y - 3 = 2 - 3 = -1$.
At $y = -3 : x = -2y - 3 = 6 - 3 = 3$.

The desired points are (−1, −1) and (3, −3).
Finally, we find the desired normals to the curve, which are the lines of slope −2 passing through each of these points. At (−1, −1), the normal line is $y = -2(x + 1) - 1$ or $y = -2x - 3$. At (3, −3), the normal line is $y = -2(x - 3) - 3$ or $y = -2x + 3$.

58.

$$x = y^2$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(y^2)$$
$$1 = 2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2y}$$

The normal line at (x, y) has slope $-2y$. Thus, the normal line at (b^2, b) is $y = -2b(x - b^2) + b$, or $y = -2bx + 2b^3 + b$.

This line intersects the x-axis at $x = \frac{2b^3 + b}{2b} = b^2 + \frac{1}{2}$, which is the value of a and must be greater than $\frac{1}{2}$ if $b \neq 0$.

The two normals at $(b^2, \pm b)$ will be perpendicular when they have slopes ± 1, which gives $-2y = \pm 1$ or $y = \pm\frac{1}{2}\left(\text{or } b = \pm\frac{1}{2}\right)$. The corresponding value of a is $b^2 + \frac{1}{2} = \left(\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}$. Thus, the two nonhorizontal normals are perpendicular when $a = \frac{3}{4}$.

59. False.

$$\frac{d}{dx}(xy^2 + x) = \frac{d}{dx}(1)$$
$$y^2 + 1 + 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-1 - y^2}{2xy}, \quad \left.\frac{dy}{dx}\right|_{(1/2,\,1)} = \frac{-1 - 1^2}{2\left(\frac{1}{2}\right)1} = -2$$

60. True. By the power rule.

$$y = (x)^{1/3}$$
$$\frac{dy}{dx} = \frac{d}{dx}(x)^{1/3} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

61. A. $\frac{d}{dx}(x^2-xy+y^2)=\frac{d}{dx}(1)$

$$2x-y+(-x+2y)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{y-2x}{2y-x}$$

62. A. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{y-2x}{2y-x}\right]$

$$=\frac{(2y-x)(y'-2)-(y-2x)(2y'-1)}{(2y-x)^2}$$

$$=\frac{(2yy'-4y-xy'+2x)-(2yy'-y-4xy'+2x)}{(2y-x)^2}$$

$$=\frac{-3y+3xy'}{(2y-x)^2}$$

$$=\frac{-3y+3x\left(\frac{y-2x}{2y-x}\right)}{(2y-x)^2}$$

$$=\frac{-6y^2+3xy+3xy-6x^2}{(2y-x)^3}$$

$$=\frac{-6(x^2-xy+y^2)}{(2y-x)^3}$$

$$=-\frac{6}{(2y-x)^3}$$

63. E. $\frac{d}{dx}(y)=\frac{d}{dx}x^{3/4}$

$$\frac{dy}{dx}=\frac{3}{4}x^{-1/4}=\frac{3}{4x^{1/4}}$$

64. C. $\frac{d}{dx}(y^2-x^2)=\frac{d}{dx}(1)$

$$-2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{x}{y}=\frac{1}{\sqrt{2}}$$

65. (a)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$b^2x^2+a^2y^2=a^2b^2$$

$$\frac{d}{dx}(b^2x^2)+\frac{d}{dx}(a^2y^2)=\frac{d}{dx}(a^2b^2)$$

$$2b^2x+2a^2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2b^2x}{2a^2y}=-\frac{b^2x}{a^2y}$$

The slope at (x_1, y_1) is $-\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$. This gives:

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$
$$a^2y_1y+b^2x_1x=a^2y_1^2+b^2x_1^2.$$

But $a^2y_1^2+b^2x_1^2=a^2b^2$ since (x_1, y_1) is on the ellipse.

Therefore, $a^2y_1y+b^2x_1x=a^2b^2$, and dividing by a^2b^2 gives $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$.

(b)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$b^2x^2-a^2y^2=a^2b^2$$

$$\frac{d}{dx}(b^2x^2)-\frac{d}{dx}(a^2y^2)=\frac{d}{dx}(a^2b^2)$$

$$2b^2x-2a^2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2b^2x}{-2a^2y}=\frac{b^2x}{a^2y}$$

The slope at (x_1,y_1) is $\frac{b^2x_1}{a^2y_1}$.

The tangent line is $y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$.

This gives:

$$a^2y_1y-a^2y_1^2=b^2x_1x-b^2x_1^2$$
$$b^2x_1^2-a^2y_1^2=b^2x_1x-a^2y_1y$$

But $b^2x_1^2-a^2y_1^2=a^2b^2$ since (x_1,y_1) is on the hyperbola. Therefore, $b^2x_1x-a^2y_1y=a^2b^2$, and dividing by a^2b^2 gives $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$.

66. (a) Solve for y:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$-\frac{y^2}{b^2}=-\frac{x^2}{a^2}+1$$

$$y^2=\frac{b^2}{a^2}(x^2-a^2)$$

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2}$$

(b) $\lim_{|x|\to\infty}\frac{f(x)}{g(x)}=\lim_{|x|\to\infty}\frac{\frac{b}{a}\sqrt{x^2-a^2}}{\frac{b}{a}|x|}$

$$=\lim_{|x|\to\infty}\frac{\sqrt{x^2-a^2}}{\sqrt{x^2}}$$

$$=\lim_{|x|\to\infty}\sqrt{1-\frac{a^2}{x^2}}=1$$

(c) $\lim_{|x|\to\infty}\frac{f(x)}{g(x)}=\lim_{|x|\to\infty}\frac{-\frac{b}{a}\sqrt{x^2-a^2}}{-\frac{b}{a}|x|}$

$$=\lim_{|x|\to\infty}\frac{\sqrt{x^2-a^2}}{\sqrt{x^2}}$$

$$=\lim_{|x|\to\infty}\sqrt{1-\frac{a^2}{x^2}}=1$$